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# Multiple Exp-Function Solutions, Group Invariant Solutions and Conservation Laws of a Generalized (2+1)-dimensional Hirota-Satsuma-Ito Equation

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## Abstract

Multiple exp-function technique and group analysis is accomplished for a comprehensive (2+1)-dimensional Hirota-Satsuma-Ito equation that appears in many sectors of nonlinear science such as for example in fluid dynamics. Travelling wave solutions are computed and it is displayed that this underlying equation gives kink solutions. The invariant reductions and further closed-form solutions are processed. Conserved currents are developed and their physical ramifications are illustrated.

**Keywords:** a generalized (2+1)-dimensional Hirota-Satsuma-Ito equation; multiple exp-function solutions; group invariant solutions; conservation laws

# 1 Introduction

Nonlinear evolution equations (NLEEs) have been used in a variety of fields, including solid state physics, fluid mechanics, plasma physics, fluid dynamics, hydrodynamics, and many others. In recent decades, many nonlinear evolution equations (NLEEs) have been studied using efficient analytical techniques to find closed-form analytic solutions [10, 11, 13]. Furthermore, using computerised symbolic computation and numerical simulation software such as Maple and Mathematica, many effective methods for solving exact solutions of NLEEs have been discovered, including the Darboux transformation, Hirota method [16, 17, 21], and Lie symmetry approach [5, 12].

One of the most versatile methods for determining the exact solutions of NLEEs is the Lie symmetry method, also known as the Lie group analysis method. The Lie symmetry technique is an effective, dependable, and robust mathematical tool in symmetry theory, with some advantages over other enormously complicated mathematical techniques used to obtain closed-form invariant solutions of nonlinear evolution equations (NLEEs) [6, 14, 15].

In 1981, Hirota and Satsuma presented a Hirota–Satsuma (HS) shallow water wave equation based on a Bäcklund transformation of the Boussinesq equation [4, 19]. It described the propagation of unidirectional shallow water waves and interactions of two long waves with different dispersion relations. Lump solutions and lump-soliton solutions were investigated in [20] and the resonant multi-soliton solutions of the HSI equation were depicted in [7]. The high-order lumps, semi-rational solutions, lump-type solutions and interaction solutions were inspected courtesy of the bilinear method [8, 9]. The non-elastic collisions between a lump wave and multi-kink waves of the (3+1)-dimensional HSI-like equation were examined in [3].

Being encouraged by the above works, we derive Lie point symmetries, conservation laws, multiple exp-function solutions, symmetry reductions and invariant solutions of a generalized (2+1)-dimensional Hirota-Satsuma-Ito equation [18]:

$$a[3\mu_t\mu_{xx} + 3\mu_x\mu_{tx} + \mu_{xxxt}] + b\mu_{ty} + c\mu_{xx} + d\mu_{xt} = 0. \tag{1.1}$$

Here $a, b, c$ and $d$ are real-valued real parameters. It should be pointed out that N-soliton solutions of (1.1) were obtained using the bilinear method and N-lump waves were constructed by applying the long wave limit to the N-solitons in [18].

# 2 Local conserved currents

Here we want to investigate local conserved currents of (1.1). Note a differential equation $E = 0$ and $\Lambda$ being the characteristic function. $\Lambda E$ is divergence if and only if $E_u(QE) = 0$, where $E_u$ is the Euler-Lagrange operator. We can now mention the following theorems.

**Theorem 2.1** *A generalized* (2+1)*-dimensional Hirota-Satsuma-Ito formally admits a unique characteristic function, namely*

$$\Lambda \;=\; f'(y)x + \left(C_1 - \frac{6f(y)a}{b}\right)u_x + g(t) + h(y), \tag{2.1}$$

*where $f(y), g(t)$ and $h(y)$ are abritary functions.*

**Proof.** A straightforward but lengthy computation from $\varepsilon_u(\Lambda E) = 0$. The expansion of this equation leads to an system of linear defining equations in the unknown distinctive function $\Lambda$.

Solving these equations, one obtains the distinctive function (2.1) □.

The existence of this characteristic function prompts the following theorem.

**Theorem 2.2** *A generalized (2+1)-dimensional Hirota-Satsuma-Ito strictly admits an infinite set of conserved currents corresponding to the unique characteristic* $\Lambda = f'(y)x+\left(C_1-\frac{6f(y)a}{b}\right)u_x+g(t)+h(y)$ *namely*

$$\begin{aligned}
T_1^t &= \frac{au_x^3}{2}+\frac{du_x^2}{4}+\frac{(2au_{xxx}+2bu_y)u_x}{8}-\frac{(uu_{xxxx}+u_{xx}^2)a}{8}-\frac{u(du_{xx}+bu_{xy})}{4},\\
T_1^x &= \frac{u(2du_{tx}+4bu_{ty})}{8}+\frac{(12u_tu_x^2+uu_{txxx}+u_tu_{xxx}-3u_{tx}u_{xx}+5u_xu_{txx})a}{8}\\
&\quad \frac{cu_x^2}{2}+\frac{du_tu_x}{4},\\
T_1^y &= -\frac{b(uu_{tx}-u_tu_x)}{4};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{4b}\Bigg(-2b^2uxf''(y)+3b\Big((u_xu_{xx}+xu_x^2+uu_x+\frac{1}{3}u_{xx})a+\frac{2bxu_y}{3}\\
&\quad -\frac{2d(u-u_{xx})}{3}\Big)f'(y)+6a\big((-2u_x^3+\frac{1}{2}uu_{xxxx}-u_xu_{xxx}+\frac{1}{2}u_{xx}^2)a\\
&\quad +b(uu_{xy}-u_xu_y)+d(uu_{xx}-u_x^2)\big)f(y)\Bigg),\\
T_2^x &= \frac{1}{4b}\Bigg(-3b\Bigg(\left((2u-3u_{xx})u_t+uu_{txx}-u_{txxx}+\frac{2u_{tx}}{3}\right)a-\frac{2dxu_t}{3}+\frac{4c(u-u_{xx})}{3}\Bigg)f'(y)\\
&\quad -6f(y)\Bigg(\Bigg(\left(6u_x^2+\frac{uxxx}{2}\right)u_t+\frac{uu_{txxx}}{2}-\frac{3u_{tx}u_{xx}}{2}+\frac{5u_xu_{txx}}{2}\Bigg)a+duu_{tx}+2buu_{ty}\\
&\quad +2cu_x^2+du_tu_x\Bigg)a\Bigg),\\
T_2^y &= \frac{bxf'(y)u_t}{2}+\frac{3af(y)(uu_{tx}-u_tu_x)}{2};
\end{aligned}$$

$$\begin{aligned}
T_3^t &= \frac{3h(y)}{4}\left(\left(uu_{xx}+u_x^2+\frac{u_{xxx}}{3}\right)a+\frac{2bu_y}{3}+\frac{2du_x}{3}\right)-\frac{bh'(y)u}{2},\\
T_3^x &= -\frac{h(y)}{4}\Big(3auu_{tx}-9au_tu_x-2du_t-3au_{txx}-4cu_x\Big),\\
T_3^y &= \frac{bh(y)u_t}{2};
\end{aligned}$$

$$
\begin{aligned}
T_4^t &= \frac{3g(t)}{4}\left(\left(uu_{xx}+u_x^2+\frac{u_{xxx}}{3}\right)a+\frac{2bu_y}{3}+\frac{2du_x}{3}\right),\\
T_4^x &= \frac{g'(t)}{4}\Big((-3uu_x-u_{xx})a-2du\Big)-\frac{3g(t)}{4}\left((uu_{tx}-3u_tu_x-u_{txx})-\frac{4cu_x}{3}-\frac{2du_t}{3}\right),\\
T_4^y &= -\frac{b(ug'(t)-u_tg(t))}{2}.
\end{aligned}
$$

**Proof.** The proof of Theorem 3.2 is straightforward but long. It consists of applying the divergence equation $\partial_t T^t+\partial_x T^x+\partial_y T^y=0$, which vanishes for all solutions of (1.1).□

We fleetingly examine the implication and physical radiance that come up from the processed conservation laws. Conservation laws inhabit in vastly critical zones both at the underpinnings of nonlinear science and its uses. Terms of bodily laws, such as conservation of energy, momentum and mass, are profoundly conservation laws. Essential physical data about the intricate comportment in nonlinear systems is kept in conservation laws. The arbitrary functions appearing in the distinct function $\Lambda$ generate basically infinite number of conserved vectors.

## 3 Lie symmetries and reductions of (1.1)

The symmetry [1] group of a generalized (2+1)-dimensional Hirota-Satsuma-Ito (gHSI) equation (1.1) will be generated by the vector field of the form

$$
X = \xi^1(t,x,y,u)\frac{\partial}{\partial t}+\xi^2(t,x,y,u)\frac{\partial}{\partial x}+\xi^3(t,x,y,u)\frac{\partial}{\partial y}+\eta(t,x,y,u)\frac{\partial}{\partial u}.
$$

**Theorem 3.1** *The Lie point symmetries are given by the following:*

$$
\begin{aligned}
&\xi^1(t,x,y,u)=\frac{3aF_3(t)}{3a-c},\\
&\xi^2(t,x,y,u)=F_1(y)+6axF_2'(y),\\
&\xi^3(t,x,y,u)=18aF_2(y),\\
&\eta(t,x,y,u)=F_4(y)+xF_5(y)-\frac{cF_3(t)}{3a-c}-(6ua+2ct)F_2'(t),
\end{aligned}
$$

*with*

$$
3aF_5(y)-bF_1'(y)+12adF_2(y)=0,
$$

*where $F_1, F_2, F_3, F_4$ and $F_5$ are arbitrary functions.*

**Proof.** Applying the fourth prolongation on equation (1.1) and thereafter splitting on the monomials of the infinitesimal coefficients and integrating the resulting overdetermined system of linear partial differential equations one obtains the above infinitesimal coefficients.

**Lemma 3.1** Suppose $F_i$ $i=1\cdots5$ are quadratic functions with respect to their arguments, then equation (1.1) admits a 12-dimensional Lie algebra spanned by the following linearly independent operators:

$$
\begin{aligned}
Z_1 &= \frac{2bxy}{a}\frac{\partial}{\partial u} + 3y^2\frac{\partial}{\partial x},\\
Z_2 &= y^2\frac{\partial}{\partial u},\\
Z_3 &= y\frac{\partial}{\partial u,}\\
Z_4 &= \frac{\partial}{\partial u},\\
Z_5 &= 3y\frac{\partial}{\partial x} + \frac{xb}{a}\frac{\partial}{\partial u},\\
Z_6 &= \frac{\partial}{\partial x},\\
Z_7 &= 12axy\frac{\partial}{\partial x} + 18ay^2\frac{\partial}{\partial y} + (2bx^2 - 12auy - 4cty - dxy)\frac{\partial}{\partial u},\\
Z_8 &= 6ax\frac{\partial}{\partial x} + 18ay\frac{\partial}{\partial y} - (6ua + 2ct + 4dx)\frac{\partial}{\partial u},\\
Z_9 &= a\frac{\partial}{\partial y},\\
Z_{10} &= \frac{3at^2}{(3a-c)}\frac{\partial}{\partial t} - \frac{ct^2}{(3a-c)}\frac{\partial}{\partial u},\\
Z_{11} &= \frac{3at}{(3a-c)}\frac{\partial}{\partial t} - \frac{ct}{(3a-c)}\frac{\partial}{\partial u},\\
Z_{12} &= \frac{3a}{(3a-c)}\frac{\partial}{\partial t} - \frac{c}{(3a-c)}\frac{\partial}{\partial u}.
\end{aligned}
$$

### 3.1 Symmetry reductions and invariant solutions of (1.1)

We aim to obtain symmetry reduction and invariant solutions of (1.1) using the symmetries we obtained above. To obtain symmetry reductions, one has to solve the associated Lagrange equations

$$\frac{dt}{\xi^1(t,x,y,u)} = \frac{dx}{\xi^2(t,x,y,u)} = \frac{dy}{\xi^3(t,x,y,u)} = \frac{du}{\eta(t,x,y,u)}.$$

We consider the following cases:

**Case 1.** We make use of the symmetry $Z_1$

$$\frac{dt}{0} = \frac{dx}{3y^2} = \frac{dy}{0} = \frac{du}{2bxy/a}$$

which gives rise to the group invariant

$$f = y,\ r = t,\ \theta = \frac{1}{3}\left(\frac{3uay - bx^2}{ya}\right).$$

Considering $f, r$ as the new independent variables and $\theta$ as the new dependent variable, equation (1.1) transforms to

$$3af\theta_{rf} + 6a\theta_r + 2c = 0 \tag{3.1}$$

which gives the general solution of the form

$$\theta(f,r) = -\frac{1}{3}\frac{cr}{a} + g(f) + f^{-2}H(r), \tag{3.2}$$

where $g(f)$ and $H(r)$ are arbitrary functions. Rewriting the solution in our original variables, the solution of gHSI takes the form

$$u(t,x,y) = \frac{1}{3}\left(\frac{3g(y)ay^2 + byx^2 - cty^2 + 3H(t)a}{ay^2}\right).$$

**Case 2.** We consider symmetry $Z_5$

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{0} = \frac{du}{xy/a}$$

that leads to

$$f = y,\ r = t,\ \theta = \frac{1}{6}\left(\frac{6uay - bx^2}{ya}\right).$$

By treating $\theta$ as the new dependent variable and $f, r$ as new independent variables, equation (1.1) transforms to

$$3af\theta_{rf} + 6a\theta_r + 2c = 0. \tag{3.3}$$

The general solution of the above equation is

$$\theta(f,r) = -\frac{1}{3}\frac{cr}{a} + g(f) + f^{-1}H(r), \tag{3.4}$$

where $g(f)$ and $H(r)$ are arbitrary functions. Consequently we have

$$u(t,x,y) = \frac{1}{6}\left(\frac{6g(y)ay^2 + byx^2 - 2cty^2 + 6H(t)a}{ay}\right).$$

**Case 3.** Symmetry $Z_7$ gives rise to

$$f = t,\ r = \frac{y}{x^{3/2}},\ \ \theta = \frac{x}{9}\left(\frac{9uay - bx^2 + 3cty + 3dxy}{ya}\right).$$

Taking $f, r$ as the new independent variables and $\theta$ as the new dependent variable, equation(1.1) transforms to

$$\begin{aligned}&9r^3\theta_{rrrf} - 18r^2\theta_{rr}\theta_f - 18r^2\theta_r\theta_{rf} + 63r^2\theta_{rrf} - 12r\theta\theta_{rf} - 66r\theta_r\theta_f + 89r\theta_{rf}\\&-24\theta\theta_f + 16\theta_f = 0.\end{aligned} \tag{3.5}$$

We now find Lie point symmetries of the above equation and use them to reduce it to an ordinary differential equation (ODE). The above equation has four symmetries:

$$\begin{aligned}S_1 &= r\frac{\partial}{\partial r},\\ S_2 &= 3r^{5/3}\frac{\partial}{\partial r} - 2\theta r^{2/3}\frac{\partial}{\partial \theta},\\ S_3 &= r^{-2/3}\frac{\partial}{\partial \theta},\\ S_{F_5(f)} &= F_5(f)\frac{\partial}{\partial f}.\end{aligned}$$

The linear combination of symmetries $S_1 + S_3$, yields

$$k = f,\ \psi = \frac{1}{2}\left(\frac{2\theta r^{2/3} + 3}{r^{2/3}}\right)$$

which gives rise to a group invariant solution $\psi = \psi(k)$ and consequently using these invariants, (3.7) is transformed into a first-order linear ordinary differential equation

$$\psi'(k) = 0$$

which has the solution

$$\psi(k) = C_1,$$

where $C_1$ is an arbitrary constant of integration. The desired solution is now

$$u(x,y,t) \quad = \quad \frac{1}{18ayx}\left[2bx^3 - 6ctxy - 6dyx^2 + 18C_1ay - 27ay\left(\frac{y}{x^{3/2}}\right)^{-2/3}\right].$$

**Case 4.** Symmetry $Z_8$ gives rise to the invariants

$$f = t,\ r = \frac{x}{y^{1/3}},\ \ \theta = \frac{1}{3}\left(\frac{(3ua + ct + dx)y^{1/3}}{a}\right).$$

Upon considering $f, r$ as the new independent variables and $\theta$ as the new dependent variable, equation(1.1) transforms to

$$br\theta_{rf} - 9a\theta_{rr}\theta_f - 9a\theta_r\theta_{rf} - 3a\theta_{rrrf} + b\theta_f = 0. \tag{3.6}$$

The above equation has the following four symmetries:

$$J_1 = 6ar\frac{\partial}{\partial r} + \left(br^2 - 6a\theta\right)\frac{\partial}{\partial \theta},$$
$$J_2 = 9a\frac{\partial}{\partial r} + rb\frac{\partial}{\partial \theta},$$
$$J_3 = \frac{\partial}{\partial \theta},$$
$$J_{G_5(f)} = G_5(f)\frac{\partial}{\partial f}.$$

The symmetry $J_1$ leads to

$$k = f,\ \phi = \frac{r}{18}\left(\frac{18a\theta - br^2}{a}\right).$$

This gives rise to a group invariant solution $\phi = \phi(k)$ and consequently using the above invariants one gets a first-order linear ordinary differential equation

$$\phi'(k) = 0,$$

which has the solution

$$\phi(k) = C_1,$$

where $C_1$ is an arbitrary constant of integration. Finally reverting back to the original variables, we obtain

$$u(x,y,t) = \frac{bx^3 - 6ctxy - 6dyx^2 + 18C_1}{18ay}.$$

**Case 5.** Taking symmetries $Z_2 + Z_6 + Z_9$ gives rise to the group invariant

$$f = t,\ r = \frac{ax-y}{a},\ \theta = \frac{1}{3}\left(\frac{3uay - y^3}{a}\right).$$

Considering $f, r$ as the new independent variables and $\theta$ as the new dependent variable, equation (1.1) transforms to

$$3a^2\theta_f\theta_{rr} + 3a^2\theta_r\theta_{rf} + ad\theta_{rf} + ca\theta_{rr} + a^2\theta_{rrrf} - b\theta_{rf} = 0 \tag{3.7}$$

which gives a solution of the form

$$\theta(f,r) = 2C_3 \tanh\left(C_3 r + C_1 - \frac{acfC_3}{4C_3^2a^2 + ad - b}\right) + C_4. \tag{3.8}$$

Rewriting the solution in our original variables we have

$$u(t,x,y) = \frac{6aC_3\tanh\left(\frac{C_3(ax-y)}{a} + C_1 - \frac{actC_3}{4C_3^2a^2+ad-b}\right) + 3aC_4 + y^3}{3a}.$$

In many applications, group invariant solutions capture the limiting behaviour of problems that far away from their initial or boundary conditions.

## 4 Multiple exp-function method

Steps of the multiple exp-function method [2] are as follows :

***Step 1. Defining solvable differential equations***

Consider a scalar $(1+1)$-dimensional partial differential equation,

$$P(x,t,u_x,u_t,\cdots) = 0. \tag{4.1}$$

We introduce a sequence of new variables $\eta_i = \eta_i(x,t)$, $1 \leqslant i \leqslant n$, by solvable PDEs, for instance, the linear ones,

$$\eta_{i,x} = k_i\eta_i, \quad \eta_{i,t} = -\omega_i\eta_i, \quad 1 \leqslant i \leqslant n, \tag{4.2}$$

where $k_i$, $1 \leqslant i \leqslant n$, are the angular wave numbers and $\omega_i$, $1 \leqslant i \leqslant n$, are the wave frequencies. It should be noted that this is often the initiating step for constructing exact solutions to nonlinear partial differential equations and also solving such linear equations leads to the exponential function solutions,

$$\eta_i = c_ie^{\xi_i}, \quad \xi_i = k_ix - \omega_it, \quad 1 \leqslant i \leqslant n, \tag{4.3}$$

where $c_i$, $1 \leqslant i \leqslant n$, arbitrary constants.

***Step 2. Transforming nonlinear PDEs***

Consider rational solutions in the new variables $\eta_i$, $1 \leqslant i \leqslant n$:

$$u(x,t) = \frac{p(\eta_1, \eta_2, \cdots, \eta_n)}{q(\eta_1, \eta_2, \cdots, \eta_n)}, \quad p = \sum_{r,s=1}^{n} \sum_{i,j=0}^{M} p_{rs,ij} \eta i_r \eta j_s,$$

$$q = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} q_{rs,ij} \eta i_r \eta j_s, \tag{4.4}$$

where $p_{kl,i,j}$ and $q_{kl,i,j}$ are all constants to be determined from the original equation (4.1). By manipulating differential relations in (4.2), we can express all partial derivatives of $u$ with $x$ and $t$ in terms of $\eta_i$, $1 \leqslant i \leqslant n$. For example, we can have

$$\begin{aligned} u_t &= \frac{q \sum_{i=1}^{n} p_{\eta_i} \eta_{i,t} - p \sum_{i=1}^{n} q_{\eta_i} \eta_{i,t}}{q^2} \\ &= \frac{-q \sum_{i=1}^{n} \omega_i p_{\eta_i} \eta_i + p \sum_{i=1}^{n} \omega_i q_{\eta_i} \eta_i}{q^2} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} u_x &= \frac{q \sum_{i=1}^{n} p_{\eta_i} \eta_{i,x} - p \sum_{i=1}^{n} q_{\eta_i} \eta_{i,x}}{q^2} \\ &= \frac{q \sum_{i=1}^{n} k_i p_{\eta_i} \eta_i - p \sum_{i=1}^{n} k_i q_{\eta_i} \eta_i}{q^2} \end{aligned} \tag{4.6}$$

where $p_{\eta_i}$ and $q_{\eta_i}$ are partial derivatives of $p$ and $q$ with respect to $\eta_i$. Substituting (4.4) and its derivatives leads to rational function equation in the new variables $\eta_i$, $1 \leqslant i \leqslant n$

$$Q(x, t, \eta_1, \eta_2, \cdots, \eta_n) = 0. \tag{4.7}$$

This is called the transformed equation of the original equation (4.1).

***Step 3. Solving algebraic systems***

Now we set the numerator of the resulting rational function $Q(x, t, \eta_1, \eta_2, \cdots, \eta_n)$ to zero. This yields a system of algebraic equations of all variables $k_i, \omega_i, p_{kl,ij}, q_{kl,ij}$. We solve this system to determine two polynomials $p$ and $q$ and the wave exponents $\xi_i$, $1 \leqslant i \leqslant n$. As a result, the multiple wave solution $u$ is computed and given by

$$u(x,t) = \frac{p(c_1 e^{k_1 x - \omega_1 t}, \cdots, c_n e^{k_n x - \omega_n t})}{q(c_1 e^{k_1 x - \omega_1 t}, \cdots, c_n e^{k_n x - \omega_n t})}. \tag{4.8}$$

### 4.1 Application of the multiple exp-function method to the generalized (2+1)-dimensional Hirota-Satsuma-Ito (gHSI) equation (1.1)

In this subsection, we will employ the multiple exp-function method to obtain one-, two- and three-wave solutions of (1.1).

#### 4.1.1 One-wave solution of (1.1)

We start with one-wave function (4.9)

$$u(x,t,y) = \frac{p}{q}, \;\; p = A_1 e^z, \;\; q = 1 + e^z \tag{4.9}$$

where $A_1$ is a constant and

$$z = k_1 x + l_1 y - \omega_1 t. \tag{4.10}$$

By applying the multiple exp-function algorithm, we obtain with the aid of Maple:

$$A_1 = 2k_1, \;\; \omega_1 = \frac{ck_1^2}{ak_1^3 + bl_1 + dk_1}. \tag{4.11}$$

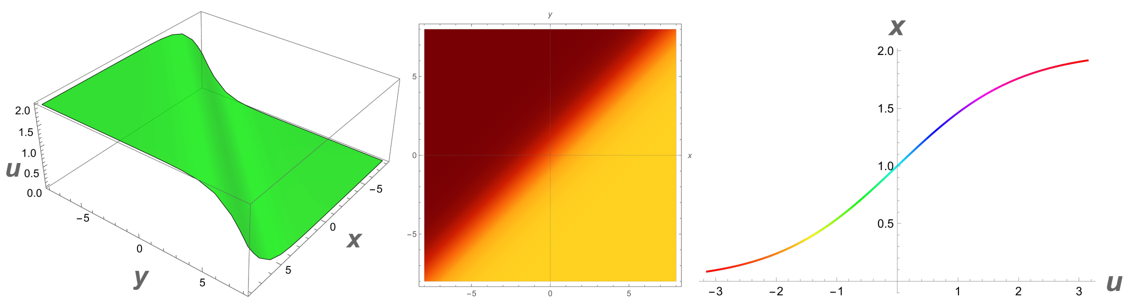


Figure 1: Evolution of the one-wave solution (4.9).

#### 4.1.2 Two-wave solution of (1.1)

We consider two-wave solution (4.12):

$$u(x,t,y) = \frac{p}{q}, \tag{4.12}$$

with $p$ and $q$ being defined by

$$\begin{aligned} p &= 2\,k_1 \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + 2\,k_2 \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \\ &\quad + 2\,A_{12}\,(k_1 + k_2)\,\mathrm{e}^{k_1 x + l_1 y - \omega_1 t}\mathrm{e}^{k_2 x + l_2 y - \omega_2 t}, \end{aligned} \tag{4.13}$$

$$q = 1 + \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} + A_{12}\mathrm{e}^{k_1 x + l_1 y - \omega_1 t}\mathrm{e}^{k_2 x + l_2 y - \omega_2 t}. \tag{4.14}$$

Applying the multiple exp-function algorithm, with the aid of Maple leads to the following three cases:

**Case 1.**

$$c = -\frac{\omega_2\left(a{k_1}^3 k_2 - \left(a{k_2}^3 + bl_2\right)k_1 + bk_2 l_1\right)J}{3{k_1}^2 a{k_2}^3\left(\left(A_{12} - 1\right){k_1}^2 + 2\,k_2\left(A_{12} + 1\right)k_1 + {k_2}^2\left(A_{12} - 1\right)\right)},$$

$$\omega_1 = -\frac{k_1\omega_2 J}{2k_2\left((A_{12}-1)\,ak_2{k_1}^3+\frac{3ak_2{}^2(A_{12}+1)k_1{}^2}{2}+\frac{(ak_2{}^3+bl_2)k_1}{2}-\frac{k_2bl_1}{2}\right)},$$

$$d = \frac{M}{3{k_1}^2a{k_2}^2\left((A_{12}-1)\,{k_1}^2+2\,k_2\,(A_{12}+1)\,k_1+{k_2}^2\,(A_{12}-1)\right)},$$

$$\begin{aligned} J &= \left((A_{12}-1)\left(ak_2{k_1}^3+2\left(a{k_2}^3-\frac{bl_2}{2}\right)k_1+k_2bl_1\right)+3\,a{k_2}^2\,(A_{12}+1)\,{k_1}^2\right),\\ M &= -a^2{k_2}^2\,(A_{12}-1)\,{k_1}^6-3\,a^2{k_2}^3\,(A_{12}+1)\,{k_1}^5-4\,(A_{12}-1)\left(a{k_2}^3+1/4\,bl_2\right)ak_2{k_1}^4\\ &\quad -3\,a{k_2}^2\left(a\,(A_{12}+1)\,{k_2}^3+1/3\,(2\,(l_1+3/2\,l_2)\,A_{12}-2\,l_1+3\,l_2)\,b\right){k_1}^3\\ &\quad +\big(-a^2\,(A_{12}-1)\,{k_2}^6-3\,((l_1+2/3\,l_2)\,A_{12}+l_1-2/3\,l_2)\,ab{k_2}^3\\ &\quad -b^2{l_2}^2\,(A_{12}-1)\big){k_1}^2-k_2bl_1\,(A_{12}-1)\left(a{k_2}^3-2\,bl_2\right)k_1-b^2{k_2}^2{l_1}^2\,(A_{12}-1)\,. \end{aligned}$$

**Case 2.**

$$\begin{aligned} a &= \frac{b\,(k_1l_2-k_2l_1)}{{k_1}^3k_2-k_1{k_2}^3},\\ c &= \frac{\omega_1\left(b{k_1}^3l_2-b{k_2}^3l_1+d{k_1}^3k_2-dk_1{k_2}^3\right)}{k_2{k_1}^2\left({k_1}^2-{k_2}^2\right)},\\ A_{12} &= \frac{(k_1-k_2)^2}{(k_1+k_2)^2},\quad \omega_2=\frac{k_2\omega_1}{k_1}. \end{aligned}$$

**Case 3.**

$$\begin{aligned} a &= -2\,\frac{b\,(k_1l_2-k_2l_1)}{{k_1}^3k_2-k_1{k_2}^3},\quad c=-3\,\frac{(k_1l_2-k_2l_1)\,\omega_1b\omega_2}{k_2k_1\,(k_1\omega_2-k_2\omega_1)},\\ d &= -\frac{b\left({k_1}^4l_2\omega_2+2\,{k_1}^3k_2l_2\omega_1-3\,{k_2}^2\,(l_1\omega_1+l_2\omega_2)\,{k_1}^2+2\,k_1{k_2}^3l_1\omega_2+{k_2}^4l_1\omega_1\right)}{\left({k_1}^2-{k_2}^2\right)k_1k_2\,(k_1\omega_2-k_2\omega_1)},\\ A_{12} &= \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \end{aligned}$$

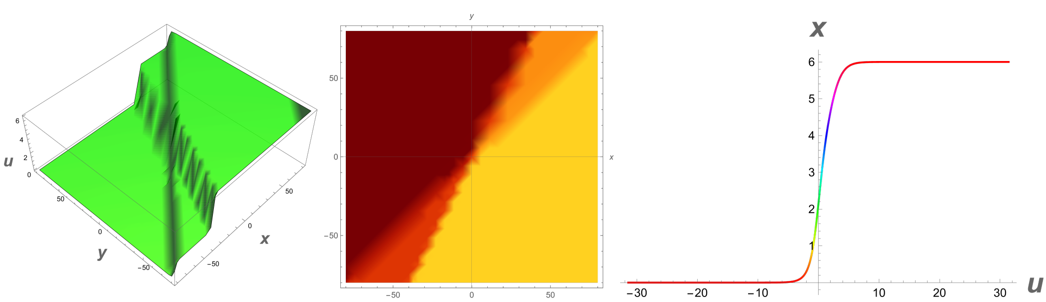

Figure 2: Evolution of the two-wave solution (4.12).

#### 4.1.3 Three-wave solution of (1.1)

For a three-wave solution one can choose:

$$u(x,t,y) = \frac{p}{q}, \tag{4.15}$$

with $p$ and $q$ being defined by

$$\begin{aligned} p &= 2\,k_1 \mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} + 2\,k_2 \mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} + 2\,k_3 \mathrm{e}^{-\omega_3 t + k_3 x + l_3 y} \\ &\quad + 2\,A_{12}\,(k_1 + k_2)\,\mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} \mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} \\ &\quad + 2\,A_{13}\,(k_1 + k_3)\,\mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} \mathrm{e}^{-\omega_3 t + k_3 x + l_3 y} \\ &\quad + 2\,A_{23}\,(k_2 + k_3)\,\mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} \mathrm{e}^{-\omega_3 t + k_3 x + l_3 y} \\ &\quad + 2\,A_{12} A_{13} A_{23}\,(k_1 + k_2 + k_3)\,\mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} \mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} \mathrm{e}^{-\omega_3 t + k_3 x + l_3 y}, \\ q &= 1 + \mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} + \mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} + \mathrm{e}^{-\omega_3 t + k_3 x + l_3 y} + A_{12} \mathrm{e}^{-\omega_1 t + k_1 x + l_1 y} \mathrm{e}^{-\omega_2 t + k_2 x + l_2 y} \\ &\quad + A_{13} \mathrm{e}^{-\omega_1 t + k_1 x + y l_1} \mathrm{e}^{-t\omega_3 + x k_3 + y l_3} + A_{23} \mathrm{e}^{-t\omega_2 + x k_2 + y l_2} \mathrm{e}^{-t\omega_3 + x k_3 + y l_3} \\ &\quad + A_{12} A_{13} A_{23} \mathrm{e}^{-\omega_1 t + k_1 x + y l_1} \mathrm{e}^{-t\omega_2 + x k_2 + y l_2} \mathrm{e}^{-t\omega_3 + x k_3 + y l_3}. \end{aligned}$$

Applying the multiple exp-function algorithm with the aid of Maple, leads to the following five cases:

**Case 1.**

$$\begin{aligned} a &= \frac{-2b\,(k_1 l_2 - k_2 l_1)}{{k_1}^3 k_2 - k_1 {k_2}^3}, \\ c &= \frac{3b\omega_2\,(k_1 l_2 - k_2 l_1)}{2k_1 {k_2}^2}, \\ d &= \frac{b\left({k_1}^3 l_2 - 3\,{k_1}^2 k_2 l_1 + 3\,k_1 {k_2}^2 l_2 - {k_2}^3 l_1\right)}{2\,{k_1}^3 k_2 - 2\,k_1 {k_2}^3}, \\ A_{13} &= \frac{T}{W}, \quad A_{23} = \frac{D}{F}, \quad \omega_1 = -\frac{k_1 \omega_2}{k_2}, \\ \omega_3 &= \frac{3{k_3}^2 \omega_2\,(k_1 - k_2)\,(k_1 + k_2)\,(k_1 l_2 - k_2 l_1)}{S}, \end{aligned}$$

$$\begin{aligned} T &= \Big( (k_2 l_3 - k_3 l_2)\,{k_1}^3 - 3\,l_2 k_3\,(-k_3 + k_2)\,{k_1}^2 + \big( -2\,{k_3}^3 l_2 - 3\,k_2\,(l_1 - l_2)\,{k_3}^2 \\ &\quad + 3\,{k_2}^2 k_3 l_1 - {k_2}^3 l_3 \big) k_1 + k_2 k_3 l_1\,(-k_3 + k_2)\,(k_2 - 2\,k_3) \Big) \Big( (k_2 l_3 - k_3 l_2)\,{k_1}^3 \\ &\quad + 3\,l_2 k_3\,(k_2 + k_3)\,{k_1}^2 + \left( -2\,{k_3}^3 l_2 - 3\,k_2\,(l_1 + l_2)\,{k_3}^2 - 3\,{k_2}^2 k_3 l_1 - {k_2}^3 l_3 \right) k_1 \\ &\quad + k_2 k_3 l_1\,(k_2 + 2\,k_3)\,(k_2 + k_3) \Big), \\ W &= \Big( (k_2 l_3 - k_3 l_2)\,{k_1}^3 - 3\,l_2 k_3\,(k_2 + k_3)\,{k_1}^2 + \big( -2\,{k_3}^3 l_2 + 3\,k_2\,(l_1 - l_2)\,{k_3}^2 \\ &\quad + 3\,{k_2}^2 k_3 l_1 - {k_2}^3 l_3 \big) k_1 + k_2 k_3 l_1\,(k_2 + 2\,k_3)\,(k_2 + k_3) \Big) \Big( (k_2 l_3 - k_3 l_2)\,{k_1}^3 \\ &\quad + 3\,l_2 k_3\,(-k_3 + k_2)\,{k_1}^2 + \left( -2\,{k_3}^3 l_2 + 3\,k_2\,(l_1 + l_2)\,{k_3}^2 - 3\,{k_2}^2 k_3 l_1 - {k_2}^3 l_3 \right) k_1 \\ &\quad + k_2 k_3 l_1\,(-k_3 + k_2)\,(k_2 - 2\,k_3) \Big), \\ D &= \Big( (-k_1 l_3 + k_3 l_1)\,{k_2}^3 + 3\,k_3 l_1\,(k_1 - k_3)\,{k_2}^2 + \big( 2\,{k_3}^3 l_1 - 3\,k_1\,(l_1 - l_2)\,{k_3}^2 \end{aligned}$$

$$
\begin{aligned}
&-3\,{k_1}^2k_3l_2+{k_1}^3l_3\big)k_2-k_1k_3l_2\,(k_1-k_3)\,(k_1-2\,k_3)\,\big)\big(\,(-k_1l_3+k_3l_1)\,{k_2}^3\\
&-3\,k_3l_1\,(k_1+k_3)\,{k_2}^2+\big(2\,{k_3}^3l_1+3\,k_1\,(l_1+l_2)\,{k_3}^2+3\,{k_1}^2k_3l_2+{k_1}^3l_3\big)\,k_2\\
&-k_1k_3l_2\,(k_1+2\,k_3)\,(k_1+k_3)\,\big),\\
F\ =\ &\big(\,(-k_1l_3+k_3l_1)\,{k_2}^3+3\,k_3l_1\,(k_1+k_3)\,{k_2}^2+\big(2\,{k_3}^3l_1+3\,k_1\,(l_1-l_2)\,{k_3}^2\\
&-3\,{k_1}^2k_3l_2+{k_1}^3l_3\big)k_2-k_1k_3l_2\,(k_1+2\,k_3)\,(k_1+k_3)\,\big)\big(\,(-k_1l_3+k_3l_1)\,{k_2}^3\\
&-3\,k_3l_1\,(k_1-k_3)\,{k_2}^2+\big(2\,{k_3}^3l_1-3\,k_1\,(l_1+l_2)\,{k_3}^2+3\,{k_1}^2k_3l_2+{k_1}^3l_3\big)\,k_2\\
&-k_1k_3l_2\,(k_1-k_3)\,(k_1-2\,k_3)\,\big),\\
S\ =\ &k_2\big(2\,{k_1}^3k_2l_3+{k_1}^3k_3l_2-3\,{k_1}^2k_2k_3l_1-2\,k_1{k_2}^3l_3+3\,k_1{k_2}^2k_3l_2-4\,k_1{k_3}^3l_2\\
&-{k_2}^3k_3l_1+4\,k_2{k_3}^3l_1\big).
\end{aligned}
$$

**Case 2.**

$$
\begin{aligned}
a\ &=\ -\frac{b\left(\omega_3\omega_2\,(k_2l_3-k_3l_2)\,{k_1}^2-\omega_1\left({k_2}^2l_3\omega_3-{k_3}^2l_2\omega_2\right)k_1+k_2k_3l_1\omega_1\,(k_2\omega_3-k_3\omega_2)\right)}{k_2k_1\left(\omega_1\,(k_2\omega_3-k_3\omega_2)\,{k_1}^2-\omega_3\omega_2\,(k_2-k_3)\,(k_2+k_3)\,k_1-k_2k_3\omega_1\,(k_3\omega_3-k_2\omega_2)\right)k_3},\\
c\ &=\ -\frac{\omega_3\omega_1\omega_2\left((k_2l_3-k_3l_2)\,{k_1}^3+\left(-{k_2}^3l_3+{k_3}^3l_2\right)k_1+{k_2}^3k_3l_1-k_2{k_3}^3l_1\right)b}{k_2k_1\left(\omega_1\,(k_2\omega_3-k_3\omega_2)\,{k_1}^2-\omega_3\omega_2\,(k_2-k_3)\,(k_2+k_3)\,k_1-k_2k_3\omega_1\,(k_3\omega_3-k_2\omega_2)\right)k_3},\\
d\ &=\ -\frac{b\left(\omega_1\left({k_2}^2l_3\omega_3-{k_3}^2l_2\omega_2\right){k_1}^3-\omega_3\omega_2\left({k_2}^3l_3-{k_3}^3l_2\right){k_1}^2-{k_2}^2{k_3}^2l_1\omega_1\,(-k_2\omega_2+k_3\omega_3)\right)}{k_2k_1\left(\omega_1\,(k_2\omega_3-k_3\omega_2)\,{k_1}^2-\omega_3\omega_2\,(k_2-k_3)\,(k_2+k_3)\,k_1-k_2k_3\omega_1\,(k_3\omega_3-k_2\omega_2)\right)k_3},\\
A_{12}\ &=\ \frac{G_1}{G_2},\quad A_{13}=\frac{E_1}{E_2},\quad A_{23}=\frac{E}{G},
\end{aligned}
$$

$$
\begin{aligned}
E\ =\ &-\omega_1{\omega_3}^2\,(k_1l_3-k_3l_1)\,{k_2}^5-k_3\omega_1\omega_3\,(\omega_2-3\,\omega_3)\,(k_1l_3-k_3l_1)\,{k_2}^4\\
&-\big(2\,l_1\omega_1\,(\omega_2-\omega_3)\,(\omega_2+\omega_3)\,{k_3}^3-k_1l_3\omega_1\left({\omega_2}^2-3\,\omega_2\omega_3+3\,{\omega_3}^2\right){k_3}^2\\
&+3\,{k_1}^2l_3\omega_2\omega_3\,(\omega_2-\omega_3)\,k_3+{k_1}^3l_3\omega_1{\omega_3}^2\big){k_2}^3-k_3\big(-3\,\omega_2\omega_1l_1\,(\omega_2-1/3\,\omega_3)\,{k_3}^3\\
&-3\,l_2\omega_1\left({\omega_2}^2-\omega_2\omega_3+1/3\,{\omega_3}^2\right)k_1{k_3}^2+3\,{k_1}^2\omega_2\omega_3\,(\omega_2-\omega_3)\,(l_2+l_3)\,k_3\\
&+{k_1}^3\omega_1\omega_3\,(l_2\omega_3+2\,l_3\omega_2)\,\big){k_2}^2+2\,{k_3}^2\omega_2\big(-1/2\,{k_3}^3l_1\omega_1\omega_2-3/2\,l_2\omega_1\,(\omega_2-1/3\,\omega_3)\,k_1{k_3}^2\\
&+3/2\,{k_1}^2l_2\omega_3\,(\omega_2-\omega_3)\,k_3+{k_1}^3\omega_1\,(l_2\omega_3+1/2\,l_3\omega_2)\,\big)k_2-k_1{k_3}^3l_2\omega_1{\omega_2}^2\,(k_1-k_3)\,(k_1+k_3)\,,\\
G\ =\ &-\omega_1{\omega_3}^2\,(k_1l_3-k_3l_1)\,{k_2}^5-k_3\omega_1\omega_3\,(\omega_2+3\,\omega_3)\,(k_1l_3-k_3l_1)\,{k_2}^4\\
&-\big(2\,l_1\omega_1\,(\omega_2-\omega_3)\,(\omega_2+\omega_3)\,{k_3}^3-k_1l_3\omega_1\left({\omega_2}^2+3\,\omega_2\omega_3+3\,{\omega_3}^2\right){k_3}^2\\
&+3\,{k_1}^2l_3\omega_2\omega_3\,(\omega_2+\omega_3)\,k_3+{k_1}^3l_3\omega_1{\omega_3}^2\big){k_2}^3-k_3\big(3\,\left(\omega_2+1/3\,\omega_3\right)\omega_2\omega_1l_1{k_3}^3\\
&-3\,l_2\left({\omega_2}^2+\omega_2\omega_3+1/3\,{\omega_3}^2\right)\omega_1k_1{k_3}^2+3\,{k_1}^2\omega_2\omega_3\,(\omega_2+\omega_3)\,(l_2-l_3)\,k_3\\
&+{k_1}^3\omega_1\omega_3\,(l_2\omega_3+2\,l_3\omega_2)\,\big){k_2}^2+2\,{k_3}^2\omega_2\big(-1/2\,{k_3}^3l_1\omega_1\omega_2+1/2\,\left(3\,\omega_2+\omega_3\right)l_2\omega_1k_1{k_3}^2\\
&-3/2\,{k_1}^2l_2\omega_3\,(\omega_2+\omega_3)\,k_3+{k_1}^3\omega_1\,(l_2\omega_3+1/2\,l_3\omega_2)\,\big)k_2-k_1{k_3}^3l_2\omega_1{\omega_2}^2\,(k_1-k_3)\,(k_1+k_3)\,,
\end{aligned}
$$

$$
\begin{aligned}
E_1\ =\ &\omega_2{\omega_3}^2\,(k_2l_3-k_3l_2)\,{k_1}^5+k_3\omega_2\omega_3\,(\omega_1-3\,\omega_3)\,(k_2l_3-k_3l_2)\,{k_1}^4\\
&+\big(2\,l_2\omega_2\,(\omega_1-\omega_3)\,(\omega_1+\omega_3)\,{k_3}^3+k_2l_3\omega_2\left({\omega_1}^2-3\,\omega_1\omega_3+3\,{\omega_3}^2\right){k_3}^2\\
&-3\,{k_2}^2l_3\omega_1\omega_3\,(\omega_1-\omega_3)\,k_3-{k_2}^3l_3\omega_2{\omega_3}^2\big){k_1}^3+k_3\big(-3\,\omega_2l_2\omega_1\,(\omega_1-1/3\,\omega_3)\,{k_3}^3\\
&-3\,\omega_2\left({\omega_1}^2-\omega_1\omega_3+1/3\,{\omega_3}^2\right)k_2l_1{k_3}^2+3\,{k_2}^2\omega_1\omega_3\,(\omega_1-\omega_3)\,(l_1+l_3)\,k_3\\
&+{k_2}^3\omega_2\omega_3\,(l_1\omega_3+2\,l_3\omega_1)\,\big){k_1}^2-2\,{k_3}^2\omega_1\big(-1/2\,{k_3}^3l_2\omega_1\omega_2
\end{aligned}
$$

$$\begin{aligned}
&-3/2\,\omega_2k_2l_1\left(\omega_1-1/3\,\omega_3\right)k_3^{\,2}+3/2\,k_2^{\,2}l_1\omega_3\left(\omega_1-\omega_3\right)k_3+\omega_2k_2^{\,3}\left(l_1\omega_3+1/2\,l_3\omega_1\right)\big)k_1\\
&+k_2k_3^{\,3}l_1\omega_1^{\,2}\omega_2\left(-k_3+k_2\right)\left(k_2+k_3\right),
\end{aligned}$$

$$\begin{aligned}
E_2 \;=\;& \omega_2\omega_3^{\,2}\left(k_2l_3-k_3l_2\right)k_1^{\,5}+k_3\omega_2\omega_3\left(\omega_1+3\,\omega_3\right)\left(k_2l_3-k_3l_2\right)k_1^{\,4}\\
&+\big(2\,l_2\omega_2\left(\omega_1-\omega_3\right)\left(\omega_1+\omega_3\right)k_3^{\,3}+k_2l_3\omega_2\left(\omega_1^{\,2}+3\,\omega_1\omega_3+3\,\omega_3^{\,2}\right)k_3^{\,2}\\
&-3\,k_2^{\,2}l_3\omega_1\omega_3\left(\omega_1+\omega_3\right)k_3-k_2^{\,3}l_3\omega_2\omega_3^{\,2}\big)k_1^{\,3}+\big(3\,\left(\omega_1+1/3\,\omega_3\right)\omega_2l_2\omega_1k_3^{\,3}\\
&-3\,\omega_2k_2l_1\left(\omega_1^{\,2}+\omega_1\omega_3+1/3\,\omega_3^{\,2}\right)k_3^{\,2}+3\,k_2^{\,2}\omega_1\omega_3\left(\omega_1+\omega_3\right)\left(l_1-l_3\right)k_3\\
&+k_2^{\,3}\omega_2\omega_3\left(l_1\omega_3+2\,l_3\omega_1\right)\big)k_3k_1^{\,2}-2\,k_3^{\,2}\omega_1\big(-1/2\,k_3^{\,3}l_2\omega_1\omega_2+1/2\,\left(3\,\omega_1+\omega_3\right)\omega_2k_2l_1k_3^{\,2}\\
&-3/2\,k_2^{\,2}l_1\omega_3\left(\omega_1+\omega_3\right)k_3+\omega_2k_2^{\,3}\left(l_1\omega_3+1/2\,l_3\omega_1\right)\big)k_1+k_2k_3^{\,3}l_1\omega_1^{\,2}\omega_2\left(-k_3+k_2\right)\left(k_2+k_3\right),
\end{aligned}$$

$$\begin{aligned}
G_1 \;=\;& \omega_2^{\,2}\omega_3\left(k_2l_3-k_3l_2\right)k_1^{\,5}+k_2\omega_2\omega_3\left(\omega_1-3\,\omega_2\right)\left(k_2l_3-k_3l_2\right)k_1^{\,4}\\
&-\big(2\,l_3\omega_3\left(\omega_1-\omega_2\right)\left(\omega_1+\omega_2\right)k_2^{\,3}-k_3l_2\omega_3\left(\omega_1^{\,2}-3\,\omega_1\omega_2+3\,\omega_2^{\,2}\right)k_2^{\,2}\\
&+3\,k_3^{\,2}l_2\omega_1\omega_2\left(\omega_1-\omega_2\right)k_2+k_3^{\,3}l_2\omega_2^{\,2}\omega_3\big)k_1^{\,3}+3\,k_2\big(l_3\omega_1\omega_3\left(\omega_1-1/3\,\omega_2\right)k_2^{\,3}\\
&+k_3l_1\omega_3\left(\omega_1^{\,2}-\omega_1\omega_2+1/3\,\omega_2^{\,2}\right)k_2^{\,2}-k_3^{\,2}\omega_1\omega_2\left(\omega_1-\omega_2\right)\left(l_1+l_2\right)k_2\\
&-1/3\,k_3^{\,3}\omega_2\omega_3\left(l_1\omega_2+2\,l_2\omega_1\right)\big)k_1^{\,2}-\omega_1\big(k_2^{\,3}l_3\omega_1\omega_3+3\,k_3l_1\omega_3\left(\omega_1-1/3\,\omega_2\right)k_2^{\,2}\\
&-3\,k_3^{\,2}l_1\omega_2\left(\omega_1-\omega_2\right)k_2-2\,k_3^{\,3}\omega_3\left(l_1\omega_2+1/2\,l_2\omega_1\right)\big)k_2^{\,2}k_1\\
&+k_2^{\,3}k_3l_1\omega_1^{\,2}\omega_3\left(-k_3+k_2\right)\left(k_2+k_3\right),\\
G_2 \;=\;& \omega_2^{\,2}\omega_3\left(k_2l_3-k_3l_2\right)k_1^{\,5}+k_2\omega_2\omega_3\left(\omega_1+3\,\omega_2\right)\left(k_2l_3-k_3l_2\right)k_1^{\,4}\\
&-\big(2\,l_3\omega_3\left(\omega_1-\omega_2\right)\left(\omega_1+\omega_2\right)k_2^{\,3}-k_3l_2\omega_3\left(\omega_1^{\,2}+3\,\omega_1\omega_2+3\,\omega_2^{\,2}\right)k_2^{\,2}\\
&+3\,k_3^{\,2}l_2\omega_1\omega_2\left(\omega_1+\omega_2\right)k_2+k_3^{\,3}l_2\omega_2^{\,2}\omega_3\big)k_1^{\,3}-3\,\big(l_3\omega_1\omega_3\left(\omega_1+1/3\,\omega_2\right)k_2^{\,3}\\
&-k_3l_1\omega_3\left(\omega_1^{\,2}+\omega_1\omega_2+1/3\,\omega_2^{\,2}\right)k_2^{\,2}+k_3^{\,2}\omega_1\omega_2\left(\omega_1+\omega_2\right)\left(l_1-l_2\right)k_2\\
&+1/3\,k_3^{\,3}\omega_2\omega_3\left(l_1\omega_2+2\,l_2\omega_1\right)\big)k_2k_1^{\,2}-\omega_1k_2^{\,2}\big(k_2^{\,3}l_3\omega_1\omega_3-3\,k_3\left(\omega_1+1/3\,\omega_2\right)l_1\omega_3k_2^{\,2}\\
&+3\,k_3^{\,2}l_1\omega_2\left(\omega_1+\omega_2\right)k_2-2\,k_3^{\,3}\omega_3\left(l_1\omega_2+1/2\,l_2\omega_1\right)\big)k_1\\
&+k_2^{\,3}k_3l_1\omega_1^{\,2}\omega_3\left(-k_3+k_2\right)\left(k_2+k_3\right).
\end{aligned}$$

**Case 3.**

$$\begin{aligned}
a &= \frac{-2b\left(k_2l_3-k_3l_2\right)}{k_2^{\,3}k_3-k_2k_3^{\,3}},\quad c=\frac{3\left(k_2l_3-k_3l_2\right)b\omega_3}{2k_2k_3^{\,2}},\\
d &= \frac{b\left(k_2^{\,3}l_3-3\,k_2^{\,2}k_3l_2+3\,k_2k_3^{\,2}l_3-k_3^{\,3}l_2\right)}{2\,k_2^{\,3}k_3-2\,k_2k_3^{\,3}},\\
A_{12} &= \frac{M_1}{M_2},\quad A_{13}=\frac{N_1}{N_2},\quad \omega_2=-\frac{k_2\omega_3}{k_3},\\
\omega_1 &= \frac{-3k_1^{\,2}\omega_3\left(k_2-k_3\right)\left(k_2+k_3\right)\left(k_2l_3-k_3l_2\right)}{H_1},
\end{aligned}$$

$$\begin{aligned}
M_1 \;=\;& \big(\left(k_2l_3-k_3l_2\right)k_1^{\,3}-1/2\,\left(-3\,k_3+3\,k_2\right)\left(k_2l_3-k_3l_2\right)k_1^{\,2}\\
&+\left(1/2\,k_2^{\,3}l_3-3/2\,k_2^{\,2}k_3l_3+3/2\,k_2k_3^{\,2}l_2-1/2\,k_3^{\,3}l_2\right)k_1\\
&-1/2\,k_2^{\,3}k_3l_1+1/2\,k_2k_3^{\,3}l_1\big)\big(\left(k_2l_3-k_3l_2\right)k_1^{\,3}-1/2\,\left(3\,k_2+3\,k_3\right)\left(k_2l_3-k_3l_2\right)k_1^{\,2}\\
&+\left(1/2\,k_2^{\,3}l_3+3/2\,k_2^{\,2}k_3l_3-3/2\,k_2k_3^{\,2}l_2-1/2\,k_3^{\,3}l_2\right)k_1-1/2\,k_2^{\,3}k_3l_1+1/2\,k_2k_3^{\,3}l_1\big),\\
M_2 \;=\;& \big(\left(k_2l_3-k_3l_2\right)k_1^{\,3}+1/2\,\left(-3\,k_3+3\,k_2\right)\left(k_2l_3-k_3l_2\right)k_1^{\,2}\\
&+\left(1/2\,k_2^{\,3}l_3-3/2\,k_2^{\,2}k_3l_3+3/2\,k_2k_3^{\,2}l_2-1/2\,k_3^{\,3}l_2\right)k_1
\end{aligned}$$

$$
\begin{aligned}
& -1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big)\big(\,(k_2l_3-k_3l_2)\,{k_1}^3+1/2\,\left(3\,k_2+3\,k_3\right)(k_2l_3-k_3l_2)\,{k_1}^2\\
& +\big(1/2\,{k_2}^3l_3+3/2\,{k_2}^2k_3l_3-3/2\,k_2{k_3}^2l_2-1/2\,{k_3}^3l_2\big)\,k_1-1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big),\\
N_1 \;=\; & \big(\,(k_2l_3-k_3l_2)\,{k_1}^3+1/2\,\left(-3\,k_3+3\,k_2\right)(k_2l_3-k_3l_2)\,{k_1}^2\\
& +\big(1/2\,{k_2}^3l_3-3/2\,{k_2}^2k_3l_3+3/2\,k_2{k_3}^2l_2-1/2\,{k_3}^3l_2\big)\,k_1\\
& -1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big)\big(\,(k_2l_3-k_3l_2)\,{k_1}^3-1/2\,\left(3\,k_2+3\,k_3\right)(k_2l_3-k_3l_2)\,{k_1}^2\\
& +\big(1/2\,{k_2}^3l_3+3/2\,{k_2}^2k_3l_3-3/2\,k_2{k_3}^2l_2-1/2\,{k_3}^3l_2\big)\,k_1-1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big),\\
N_2 \;=\; & \big(\,(k_2l_3-k_3l_2)\,{k_1}^3-1/2\,\left(-3\,k_3+3\,k_2\right)(k_2l_3-k_3l_2)\,{k_1}^2\\
& +\big(1/2\,{k_2}^3l_3-3/2\,{k_2}^2k_3l_3+3/2\,k_2{k_3}^2l_2-1/2\,{k_3}^3l_2\big)\,k_1\\
& -1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big)\big(\,(k_2l_3-k_3l_2)\,{k_1}^3+1/2\,\left(3\,k_2+3\,k_3\right)(k_2l_3-k_3l_2)\,{k_1}^2\\
& +\big(1/2\,{k_2}^3l_3+3/2\,{k_2}^2k_3l_3-3/2\,k_2{k_3}^2l_2-1/2\,{k_3}^3l_2\big)\,k_1-1/2\,{k_2}^3k_3l_1+1/2\,k_2{k_3}^3l_1\big),\\
H_1 \;=\; & k_3\big(4\,{k_1}^3k_2l_3-4\,{k_1}^3k_3l_2-k_1{k_2}^3l_3+3\,k_1{k_2}^2k_3l_2-3\,k_1k_2{k_3}^2l_3+k_1{k_3}^3l_2\\
& -2\,{k_2}^3k_3l_1+2\,k_2{k_3}^3l_1\big).
\end{aligned}
$$

**Case 4.**

$$\begin{aligned}
a &= \frac{b\left(k_2l_3-k_3l_2\right)}{{k_2}^3k_3-k_2{k_3}^3}, \quad A_{23}=\frac{\left(k_2-k_3\right)^2}{\left(k_2+k_3\right)^2},\\
c &= \frac{R_1b}{3k_2\left(k_2^2-k_3^2\right)\left(k_2l_3-k_3l_2\right)\Big[2\,k_1k_3\left(A_{13}+1\right)+\left(k_3^2+k_1^2\right)\left(A_{13}-1\right)\Big]{k_1}^2{k_3}^2},\\
d &= \frac{R_2b}{3k_2\left(k_2^2-k_3^2\right)\left(k_2l_3-k_3l_2\right)\Big[2\,k_1k_3\left(A_{13}+1\right)+\left(k_3^2+k_1^2\right)\left(A_{13}-1\right)\Big]{k_1}^2{k_3}^2},\\
A_{12} &= \frac{R_3}{R_4}, \quad \omega_1=\frac{R_5}{R_6}, \quad \omega_2=\frac{k_2\omega_3}{k_3},
\end{aligned}$$

$$\begin{aligned}
R_1 &= -\big(-2\,\left(k_1l_2+1/2\,k_2l_1\right)\left(A_{13}-1\right){k_3}^3-3\,k_1\left(l_2\left(A_{13}+1\right)k_1-k_2l_3\left(A_{13}-1\right)\right){k_3}^2\\
&\quad+\left(-l_2\left(A_{13}-1\right){k_1}^3+3\,k_2l_3\left(A_{13}+1\right){k_1}^2+{k_2}^3l_1\left(A_{13}-1\right)\right)k_3\\
&\quad+k_1k_2l_3\left(k_1-k_2\right)\left(k_1+k_2\right)\left(A_{13}-1\right)\big)\omega_3\big(\left(k_1l_2-k_2l_1\right){k_3}^3+\left(-{k_1}^3l_2+{k_2}^3l_1\right)k_3\\
&\quad+{k_1}^3k_2l_3-k_1{k_2}^3l_3\big),\\
R_2 &= -\big(\left({k_1}^2{l_2}^2+k_1k_2l_1l_2+{k_2}^2{l_1}^2\right)\left(A_{13}-1\right){k_3}^6+3\left({l_2}^2\left(A_{13}+1\right){k_1}^2+k_2l_1l_2\left(A_{13}+1\right)k_1\right.\\
&\quad\left.-{k_2}^2l_1l_3\left(A_{1,3}-1\right)\right)k_1{k_3}^5+\big(4\,{l_2}^2\left(A_{13}-1\right){k_1}^4+2\,k_2l_2\big(\left(-3/2\,A_{13}-3/2\right)l_3\\
&\quad+l_1\left(A_{13}-1\right)\big){k_1}^3-3\,{k_2}^2l_1l_3\left(A_{13}+1\right){k_1}^2-{k_2}^3l_1l_2\left(A_{13}-1\right)k_1-2\,{k_2}^4{l_1}^2\left(A_{13}-1\right)\big){k_3}^4\\
&\quad+3\left({l_2}^2\left(A_{13}+1\right){k_1}^4-7/3\,k_2l_2l_3\left(A_{13}-1\right){k_1}^3-2/3\,{k_2}^2l_1l_3\left(A_{13}-1\right){k_1}^2\right.\\
&\quad\left.-{k_2}^3l_2\left(\left(2/3\,A_{13}-2/3\right)l_3+l_1\left(A_{13}+1\right)\right)k_1+5/3\,{k_2}^4l_1l_3\left(A_{13}-1\right)\right)k_1{k_3}^3\\
&\quad+\big({l_2}^2\left(A_{13}-1\right){k_1}^6-6\,k_2l_2l_3\left(A_{13}+1\right){k_1}^5+3\,{k_2}^2{l_3}^2\left(A_{13}-1\right){k_1}^4\\
&\quad-2\,{k_2}^3l_2\left(\left(3/2\,A_{13}+3/2\right)l_3+l_1\left(A_{13}-1\right)\right){k_1}^3+3\,{k_2}^4l_1l_3\left(A_{13}+1\right){k_1}^2\\
&\quad+{k_2}^6{l_1}^2\left(A_{13}-1\right)\big){k_3}^2-2\,k_2\big(l_2\left(A_{13}-1\right){k_1}^5-3/2\,k_2l_3\left(A_{13}+1\right){k_1}^4\\
&\quad+1/2\,{k_2}^2l_2\left(A_{13}-1\right){k_1}^3-{k_2}^3\left(\left(3/2\,A_{13}+3/2\right)l_3+l_1\left(A_{13}-1\right)\right){k_1}^2\\
&\quad+{k_2}^5l_1\left(A_{13}-1\right)\big)l_3k_1k_3+{k_1}^2{k_2}^2{l_3}^2\left({k_1}^2+k_1k_2+{k_2}^2\right)\left({k_1}^2-k_1k_2+{k_2}^2\right)\left(A_{13}-1\right)\big),
\end{aligned}$$

$$\begin{aligned}
R_3 &= -\left(k_2l_3-k_3l_2\right)\left(\left(A_{13}-1\right)k_2+k_3\left(A_{13}+1\right)\right){k_1}^4\\
&\quad+2\left(-k_3+k_2\right)\left(k_2+k_3\right)\left(A_{13}-1\right)\left(k_2l_3-k_3l_2\right){k_1}^3+\big(-2\,l_3\left(A_{13}-1\right){k_2}^4\\
&\quad+k_3l_3\left(A_{13}+1\right){k_2}^3-3\,{k_3}^2\left(\left(l_2-l_3\right)A_{13}+l_2+l_3\right){k_2}^2-{k_3}^3l_2\left(A_{13}-1\right)k_2\\
&\quad+2\,{k_3}^4l_2\left(A_{13}+1\right)\big){k_1}^2+\left(k_2+k_3\right)\left(-k_3+k_2\right)\big(l_3\left(A_{13}-1\right){k_2}^3\\
&\quad+2\,k_3l_1\left(A_{13}-1\right){k_2}^2+2\,{k_3}^2l_1\left(A_{13}+1\right)k_2-{k_3}^3l_2\left(A_{13}-1\right)\big)k_1\\
&\quad-k_2k_3l_1\left(-k_3+k_2\right)^2\left(k_2+k_3\right)^2\left(A_{13}-1\right),\\
R_4 &= \left(\left(A_{13}-1\right)k_2-k_3\left(A_{13}+1\right)\right)\left(k_2l_3-k_3l_2\right){k_1}^4\\
&\quad+2\left(-k_3+k_2\right)\left(k_2+k_3\right)\left(A_{13}-1\right)\left(k_2l_3-k_3l_2\right){k_1}^3+\big(2\,l_3\left(A_{13}-1\right){k_2}^4\\
&\quad+k_3l_3\left(A_{13}+1\right){k_2}^3-3\,{k_3}^2\left(\left(l_2+l_3\right)A_{13}+l_2-l_3\right){k_2}^2+{k_3}^3l_2\left(A_{13}-1\right)k_2\\
&\quad+2\,{k_3}^4l_2\left(A_{13}+1\right)\big){k_1}^2+\left(k_2+k_3\right)\big(l_3\left(A_{13}-1\right){k_2}^3-2\,k_3l_1\left(A_{13}-1\right){k_2}^2
\end{aligned}$$

$$
\begin{aligned}
&+2\,{k_3}^2 l_1 \left(A_{13}+1\right) k_2 - {k_3}^3 l_2 \left(A_{13}-1\right)\Big) \left(-k_3+k_2\right) k_1 \\
&-k_2 k_3 l_1 \left(-k_3+k_2\right)^2 \left(k_2+k_3\right)^2 \left(A_{13}-1\right), \\
R_5 &= -\omega_3 \Big( \left(A_{13}-1\right) \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 + 3\,k_3 \left(A_{13}+1\right) \left(k_2 l_3 - k_3 l_2\right) {k_1}^2 \\
&- \left(A_{1,3}-1\right) \left({k_2}^3 l_3 - 3\,k_2 {k_3}^2 l_3 + 2\,{k_3}^3 l_2\right) k_1 + k_2 k_3 l_1 \left(-k_3+k_2\right) \left(k_2+k_3\right) \left(A_{13}-1\right)\Big) k_1, \\
R_6 &= 2k_3 \Big( \left(A_{13}-1\right) \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 + 3/2\,k_3 \left(A_{13}+1\right) \left(k_2 l_3 - k_3 l_2\right) {k_1}^2 \\
&+1/2\,\left(A_{13}-1\right) \left({k_2}^3 l_3 - {k_3}^3 l_2\right) k_1 - 1/2\,k_2 k_3 l_1 \left(-k_3+k_2\right) \left(k_2+k_3\right) \left(A_{13}-1\right)\Big).
\end{aligned}
$$

**Case 5.**

$$
\begin{aligned}
a &= -\frac{2b\left(k_1 l_3 - k_3 l_1\right)}{{k_1}^3 k_3 - k_1 {k_3}^3}, \quad c = \frac{3\left(k_1 l_3 - k_3 l_1\right) b \omega_3}{2 k_1 {k_3}^2}, \\
d &= \frac{b\left({k_1}^3 l_3 - 3\,{k_1}^2 k_3 l_1 + 3\,k_1 {k_3}^2 l_3 - {k_3}^3 l_1\right)}{2\,{k_1}^3 k_3 - 2\,k_1 {k_3}^3}, \\
A_{12} &= \frac{Z_1}{Z_2}, \quad A_{23} = \frac{Z_3}{Z_4}, \quad \omega_1 = -\frac{k_1 \omega_3}{k_3}, \\
\omega_2 &= \frac{3{k_2}^2 \omega_3 \left(k_1-k_3\right)\left(k_1+k_3\right)\left(k_1 l_3 - k_3 l_1\right)}{Z_5},
\end{aligned}
$$

$$
\begin{aligned}
Z_1 &= \Big( \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 - 3\,k_2 l_3 \left(-k_3+k_2\right) {k_1}^2 + \big(2\,{k_2}^3 l_3 + 3\,k_3 \left(l_1 - l_3\right) {k_2}^2 \\
&-3\,{k_3}^2 l_1 k_2 + {k_3}^3 l_2\big) k_1 - 2\,k_3 k_2 \left(k_2 - 1/2\,k_3\right) l_1 \left(-k_3+k_2\right)\Big) \Big( \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 \\
&-3\,k_2 l_3 \left(k_2+k_3\right) {k_1}^2 + \big(2\,{k_2}^3 l_3 + 3\,k_3 \left(l_1+l_3\right) {k_2}^2 + 3\,{k_3}^2 l_1 k_2 + {k_3}^3 l_2\big) k_1 \\
&-2\,\left(k_2+k_3\right) k_3 k_2 l_1 \left(k_2 + 1/2\,k_3\right)\Big), \\
Z_2 &= \Big( \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 + 3\,k_2 l_3 \left(-k_3+k_2\right) {k_1}^2 + \big(2\,{k_2}^3 l_3 - 3\,k_3 \left(l_1 + l_3\right) {k_2}^2 \\
&+3\,{k_3}^2 l_1 k_2 + {k_3}^3 l_2\big) k_1 - 2\,k_3 k_2 \left(k_2 - 1/2\,k_3\right) l_1 \left(-k_3+k_2\right)\Big) \Big( \left(k_2 l_3 - k_3 l_2\right) {k_1}^3 \\
&+3\,k_2 l_3 \left(k_2+k_3\right) {k_1}^2 + \big(2\,{k_2}^3 l_3 - 3\,k_3 \left(l_1 - l_3\right) {k_2}^2 - 3\,{k_3}^2 l_1 k_2 + {k_3}^3 l_2\big) k_1 \\
&-2\,\left(k_2+k_3\right) k_3 k_2 l_1 \left(k_2 + 1/2\,k_3\right)\Big), \\
Z_3 &= \Big( \left(2\,k_1 l_3 - 2\,k_3 l_1\right) {k_2}^3 + 3\,\left(k_1 - k_3\right) \left(k_1 l_3 - k_3 l_1\right) {k_2}^2 + \big({k_1}^3 l_3 - 3\,{k_1}^2 k_3 l_3 \\
&+3\,k_1 {k_3}^2 l_1 - {k_3}^3 l_1\big) k_2 - {k_1}^3 k_3 l_2 + k_1 {k_3}^3 l_2\Big) \Big( \left(2\,k_1 l_3 - 2\,k_3 l_1\right) {k_2}^3 \\
&-3\,\left(k_1 + k_3\right) \left(k_1 l_3 - k_3 l_1\right) {k_2}^2 + \big({k_1}^3 l_3 + 3\,{k_1}^2 k_3 l_3 - 3\,k_1 {k_3}^2 l_1 - {k_3}^3 l_1\big) k_2 \\
&-{k_1}^3 k_3 l_2 + k_1 {k_3}^3 l_2\Big), \\
Z_4 &= \Big( \left(2\,k_1 l_3 - 2\,k_3 l_1\right) {k_2}^3 - 3\,\left(k_1 - k_3\right) \left(k_1 l_3 - k_3 l_1\right) {k_2}^2 + \big({k_1}^3 l_3 - 3\,{k_1}^2 k_3 l_3 \\
&+3\,k_1 {k_3}^2 l_1 - {k_3}^3 l_1\big) k_2 - {k_1}^3 k_3 l_2 + k_1 {k_3}^3 l_2\Big) \Big( \left(2\,k_1 l_3 - 2\,k_3 l_1\right) {k_2}^3 \\
&+3\,\left(k_1 + k_3\right) \left(k_1 l_3 - k_3 l_1\right) {k_2}^2 + \big({k_1}^3 l_3 + 3\,{k_1}^2 k_3 l_3 - 3\,k_1 {k_3}^2 l_1 - {k_3}^3 l_1\big) k_2 \\
&-{k_1}^3 k_3 l_2 + k_1 {k_3}^3 l_2\Big), \\
Z_5 &= k_3 \Big( -\left(2\,k_1 l_2 + k_2 l_1\right) {k_3}^3 + 3\,k_1 k_2 {k_3}^2 l_3 + \big(2\,{k_1}^3 l_2 - 3\,{k_1}^2 k_2 l_1 + 4\,{k_2}^3 l_1\big) k_3 \\
&+{k_1}^3 k_2 l_3 - 4\,k_1 {k_2}^3 l_3\Big).
\end{aligned}
$$

The multiple exp-function algorithm as a generalization of Hirota's perturbation scheme has been used to construct multiple wave solutions to a the generalized (2+1)-dimensional Hirota-Satsuma-Ito equation (1.1). This paper did not show N-soliton solution since the computations are pretty complex with multiple generic phase shifts and wave frequencies.

## 5 Concluding remarks

Multiple exp-function technique and symmetry analysis was accomplished for a generalized (2+1)-dimensional Hirota-Satsuma-Ito equation which arises in many sectors of nonlinear science such as for example in fluid dynamics. Multiple wave solutions for the generalized (2+1)-dimensional Hirota-Satsuma-Ito (GHSI) equation utilizing the multiple exp-function technique were computed. It was shown that this underlying equation gave kink solutions.The similarity reductions and new exact solutions were computed. Conservation laws were derived using the multiplier approach. It should be noted that the familiarity of closed-form solutions of nonlinear ordinary and partial differential equations does indeed empower numerical solvers and assists in stability analysis. Even though many exertions have been devoted to extracting closed form solutions of nonlinear evolution equations, there is no universal method. To the best of our knowledge, this is the first time that such an endeavour was made for this underlying equation.

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